Bulgarian Training Camp for IMO 2023.

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1 Finite Diffrences, Properties.

Definition. *Having a function $f : \mathbb{R} \to \mathbb{R}$, we denote $\Delta_h^1 f(x) := f(x+h) - f(x)$, where $h \in \mathbb{R}$ is fixed. This is called a finite difference of order 1 with step h, taken at a point x. Finite differences with higher orders are defined inductively as:*

$$\Delta_h^{n+1} f(x) := \Delta_h^1 \left(\Delta_h^n f(x)\right).$$

In case $n = 1$, instead of Δ_h^1 we just write Δ_h.

It easily follows by induction an explicit equality for the n-th finite difference:

$$\Delta_h^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + ih).$$

We didn't impose any requirement on f, but in case it is "smooth" enough, $\dfrac{\Delta_h^n f(x)}{h^n}$ is close to $f^{(n)}(x)$ for small h This is justified by the following well known property.

Property 1.1. *Providing $f(x)$ is n times differentiable, there is a point $\xi \in (x, x + nh)$ such that*

$$f^{(n)}(\xi) = \frac{\Delta_h^n f(x)}{h^n}$$

For $n = 1$ it's just the Lagrange's mean value theorem, so it could be viewed as its generalization.

This resemblance of the finite difference to derivative (or rather to the differential $d^n f(x)$), poses a question. Does there exist some analogue of the Leibniz rule for the derivative of a product of two functions? Given two functions f and g, we want to represent $\Delta_h^n (fg)(x)$ as an expression of some finite differences of f and g.

Property 1.2. *Let $P(x)$ is a polynomial of degree n and $k > n$ Then*

$$\Delta_h^k [P](x) \equiv 0.$$

for any $h \in \mathbb{R}$.

Indeed, $\Delta_h P(x)$ is a polynomial of degree $n-1$ (with respect to x). So, each time the degree of the resulting polynomial decreases with 1. Hence, $\Delta_h^{n+1}[P](x) = 0$.

Property 1.3. *(Leibniz rule)*

$$\Delta_h^m (f \cdot g)(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta_h^j f(x) \cdot \Delta_h^{m-j} g(x + jh).$$

It looks exactly like the corresponding Leibniz rule for n-th derivative of two functions' product with one exception, $\Delta_h^{m-j} g$ is taken at the point $x + jh$, not at the point x.

2 Finite Differences, Applications.

Problem 2.1. *(Tuymaada 2022, Senior, p5) Prove that a quadratic trinomial $x^2 + ax + b (a, b \in R)$ cannot attain at ten consecutive integral points values equal to powers of 2 with non-negative integral exponent.*
(F. Petrov)

Solution. Let $P(x)$ be the polynomial in question and let it takes values equal to powers of 2 at the points $x = j, j+1, \ldots, j+9$. Consider the finite difference of order 2 and step 1

$$\Delta^2 P(x) := P(x) - 2P(x+1) + P(x+2)$$

Since P is a monic polynomial of degree 2, we have $\Delta^2 P(x) = 2$. On the other hand, putting $x = j, j+1, \ldots, j+9$ the longest possible sequence $f(j), f(j+1), \ldots$ of values of f all of which are powers of 2 is

$$8, 4, 2, 2, 4, 8$$

and we can only satisfy $\Delta^2 f(x) = 2$ for $x = j, j+1, j+2, j+3$. $\square$

Problem 2.2. *(USA TST 2011, p3) Let p be a prime. We say that a sequence of integers $\{z_n\}_{n=0}^{\infty}$ is a p-pod if for each integer $e \geq 0$, there is an $N \geq 0$ such that whenever $m \geq N$, p^e divides the sum $\sum_{k=0}^{m} (-1)^k \binom{m}{k} z_k$. Prove that if both sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are p-pods, then the sequence $\{x_n y_n\}_{n=0}^{\infty}$ is a p-pod.*

Solution. To stick to the above notations, let $f(n), g(n), n \in \mathbb{N}$ be the functions/sequences corresponding to $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$. We want to show that if f and g are p-pod then fg is also p-pod. The definition of the p-pod just says $\Delta_1^m f(0)$ and $\Delta_1^m g(0)$ are multiple of p^e when m is large enough. (here Δ_1^m means the m-th finite difference with step $h = 1$). Fix some $e \in \mathbb{N}$ and let m be a big number. Look at the property 2 applied to f, g for $x = 0, h = 1$.

$$\Delta_1^m(f\cdot g)(0)=\sum_{j=0}^{m}\binom{m}{j}\Delta_1^j f(0)\cdot\Delta_1^{m-j}g(j).$$

Clearly $\Delta_1^j f(0)$ is multiple of p^e whenever j is big enough. It's enough to establish that in case j is "small", then $\Delta_1^{m-j}g(j)$ is also multiple of p^e. It boils down to show the truncated sequence $g(j), g(j+1), \dots$ is also p-pod. It can be done using the definition of finite difference and induction on j. Indeed,

$$\Delta_1^m g(j-1)=\Delta_1^{m-1}g(j)-\Delta_1^{m-1}g(j-1).$$

Putting consecutively $j=1,2,\dots$ we get that $\Delta_1^{m-1}g(j)$ is multiple of p^e when m is large enough. The result follows. $\square$

Problem 2.3. *Let n be a non-negative integer. Prove that*

$$\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}i^{n+1}=\frac{n(n+1)!}{2}$$

1st solution. Both are the number of ways to paint $n+1$ balls, enumerated as $1,2,\dots,n+1$ using **all** of n different colors. LHS is counting it using inclusion-exclusion: i^{n+1} is the number of ways to paint using i selected colors (don't have to use all i colors). RHS is counting it directly: first choose a color for two balls - n ways, then there are $\frac{(n+1)n}{2}$ ways to choose these balls. The rest can be permuted in $(n-1)!$ ways. $\square$

2nd solution. Note that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}i^{n+1}=\Delta_1^n[f](0)$$

for $f(x):=x^{n+1}$ where $\Delta_h^n[f](t)$ means [url=https://dgrozev.wordpress.com/2019/09/15/finite-differences-in-olympiads/]the finite difference[/url] of order n and step h of a function f taken at a point t. To calculate this we use the following 2 properties of finite differences.

$$\Delta_h^n[x^n](t)=n! \qquad (1)$$

and for any polynomial P of degree $n-1$

$$\Delta_h^n[P](t)=0 \qquad (2)$$

We have

$$\begin{aligned}
\sum_{i=1}^{n}\binom{n}{i}(-1)^{n-i}i^{n+1} &= n\sum_{i=1}^{n}\binom{n-1}{i-1}(-1)^{n-i}i^{n}\\
&= n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}(i+1)^{n}\\
&= n\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n}+n^2\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n-1}\\
&= n\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{n-1-i}i^{n}+n^2(n-1)!
\end{aligned}$$

where in the last 2 lines we used (1) and (2). Denoting $I_n := \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}i^{n+1}$, the above chain yields

$$I_n = nI_{n-1}+n^2(n-1)! = n(I_{n-1}+n!)$$

Since $I_1 = 1$ it easily follows (e.g., by induction) $I_n = \frac{n(n+1)!}{2}$. □

Problem 2.4. *($n-1$ finite difference of x^n)*

$$\Delta^{n-1}x^n = n!x + n!\frac{n-1}{2}.$$

Proof. Here, it's the forward difference with step 1. The idea is to represent x^n as a linear combination of the basis polynomials $(x)_k := x(x-1)(x-2)\cdots(x-k+1), k = 0, 1, \ldots, n$ and use a known fact the finite differences of $(x)_k$ are easily calculated. Namely

$$\Delta^r(x)_k = k(k-1)\cdots(k-r+1)(x)_{k-r} \qquad (*).$$

It easily follows $x^n = (x)_n + \frac{n(n-1)}{2}(x)_{n-1} + P_{n-2}(x)$, where $P_{n-2}(x)$ is a polynomial of degree $n-2$. Using $(*)$ and $\Delta^{n-1}P_{n-2}(x) = 0$, we get

$$\Delta^{n-1}x^n = n!x + n!\frac{n-1}{2}.$$

□

Problem 2.5. *(4463, Crux 2019(7)) For all integers $n > m \geq 0$, prove that:*

$$\sum_{k=0}^{n}(-1)^k\cdot\tbinom{2n+1}{n-k}\cdot(2k+1)^{2m+1} = 0.$$

Solution. The idea to consider the function $P(x) := (2n+1-2x)^{2m+1}$. Its finite difference $\Delta_h^{2n+1}[P](x)$ vanishes, because P is a polynomial of degree less than $2n+1$. We will exploit it and put $h = 1$ and $x = 0$. The obtained expression will lead us to the desired identity. I suppose a question arises: how did I come up with that polynomial P? Well, the problem's identity resembles a finite difference of something, so I tried several times to guess what that "something" would be, and finally got it. Let's make now some calculations. Using (2) we get

$$0 = \Delta_1^{2n+1}[P](0) = \sum_{k=0}^{2n+1} (-1)^{2n+1-k} \binom{2n+1}{k} (2n+1-2k)^{2m+1}.$$

We partition the above sum into two parts: 1) The summation index runs from $k = 0$ to n - denote the corresponding sum by S_1; and 2) when $k = n+1$ to $2n+1$ - denote that sum by S_2. For the first sum we make a substitution $\ell = n - k$ and it yields

$$S_1 = \sum_{\ell=0}^{n} (-1)^{n-\ell-1} \binom{2n+1}{n-\ell} (2\ell+1)^{2m+1}.$$

For the second sum S_2 we set $\ell = k - 1 - n$. When k runs from $n+1$ to $2n+1$, ℓ varies from 0 to n. Thus, we get

$$S_2 = \sum_{\ell=0}^{n} (-1)^{n-\ell+1} \binom{2n+1}{n-\ell} (2\ell+1)^{2m+1}.$$

As we see, $S_1 = S_2$ and $S_1 + S_2 = 0$. It gives $S_1 = S_2 = 0$ and the result follows. □

Problem 2.6. *Given a function $f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that satisfies the following condition*

$$f(x,y) = f(x+1,y) + f(x,y+1)\,;\, f(0,0) = 1$$

prove that,

$$\big(f(x,0)\big)^2 \leq f(2x,0), \forall x \in \mathbb{Z}_{\geq 0}$$

Solution. Clearly, $0 \leq f(x,y) \leq 1, \forall x, y \in \mathbb{Z}_{\geq 0}$. Let us fix $n \in \mathbb{Z}_{\geq 0}$. Applying n times the recursive formula, starting from $f(0,0)$, we get

$$1 = f(0,0) = \sum_{i=0}^{n} \binom{n}{i} f(n-i,i). \qquad (1)$$

Note that (1) also implies that $f(n-i,i) \to 0$ as $n \to \infty$. Let us denote,

$$\Delta^i f(x,y) := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} f(x+j,y).$$

We have

$$\begin{aligned}
f(n-i,i) &= f(n-i,i-1) - f(n-i+1,i-1) = \\
&= \big(f(n-i,i-2) - f(n-i+1,i-2)\big) \\
&\quad - \big(f(n-i+1,i-2) - f(n-i+2,i-2)\big) \\
&= f(n-i,i-2) - 2f(n-i+1,i-2) + f(n-i+2,i-2) \\
&= \Delta^2 f(n-i,i-2)
\end{aligned}$$

In the same way, it can be proven by induction,

$$f(n-i,i) = (-1)^i \Delta^i f(n-i,0). \qquad (2)$$

Putting it into (1) we get

$$f(0,0) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0). \qquad (3)$$

Analogously,

$$\begin{aligned}
f(k,0) &= \sum_{i=k}^{n} \binom{n-k}{i-k} (-1)^{n-i} \Delta^{n-i} f(i,0) \\
&= \sum_{i=k}^{n} \frac{i(i-1)\cdots(i-k+1)}{n(n-1)\cdots(n-k+1)} \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0).
\end{aligned}$$

Clearly,

$$\left(\frac{i-k+1}{n-k+1}\right)^k \le \frac{i(i-1)\cdots(i-k+1)}{n(n-1)\cdots(n-k+1)} \le \left(\frac{i}{n}\right)^k. \qquad (4)$$

Let us denote,

$$g(k,n) := \sum_{i=k}^{n} \left(\frac{i}{n}\right)^k \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0).$$

From (4) we get

$$|f(k,0) - g(k,n)| \le \sum_{i=k}^{n} \left| \left(\frac{i}{n}\right)^k - \left(\frac{i-k+1}{n-k+1}\right)^k \right| \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0)$$

because $(-1)^{n-i}\Delta f(i,0) \ge 0$, which follows from (2). Therefore,

$$|f(k,0) - g(k,n)| \le \max_{i,k \le i \le n} \left| \left(\frac{i}{n}\right)^k - \left(\frac{i-k+1}{n-k+1}\right)^k \right| \cdot \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0).$$

By (3), the sum in the right hand side of the above inequality is equal to $f(0,0) = 1$. Hence,

$$|f(k,0) - g(k,n)| \le \max_{i,k\le i\le n} \left| \left(\frac{i}{n}\right)^k - \left(\frac{i-k+1}{n-k+1}\right)^k \right|.$$

Note that

$$\lim_{n\to\infty} \max_{i,k\le i\le n} \left| \left(\frac{i}{n}\right)^k - \left(\frac{i-k+1}{n-k+1}\right)^k \right| = 0.$$

Thus,

$$\begin{aligned} f(k,0) &= \lim_{n\to\infty} \sum_{i=k}^{n} \left(\frac{i}{n}\right)^k \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{i}{n}\right)^k \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0). \end{aligned}$$

The last equality holds because for $i \ne 0$, $(-1)^{n-i}\Delta^{n-i} f(i,0) = f(i, n-i) \to 0$ as $n \to \infty$. Let us denote $A_i := \binom{n}{i}(-1)^{n-i}\Delta^{n-i} f(i,0)$. By Cauchy-Schwartz inequality and (1), it yields,

$$\begin{aligned} f(k,0)^2 &= \lim_{n\to\infty} \left(\sum_{i=0}^{n} \left(\frac{i}{n}\right)^k A_i^{1/2} A_i^{1/2} \right)^2 \\ &\le \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{i}{n}\right)^{2k} \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0) \cdot \lim_{n\to\infty} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \Delta^{n-i} f(i,0) \\ &= f(2k,0) f(0,0) = f(2k,0). \end{aligned}$$

Finally, $f(k,0)^2 \le f(2k,0)$.

$\square$

3 Chebyshev Theorem. Chebyshev polynomials.

Theorem 3.1. *(Chebyshev's equioscillation theorem) Let $f(x)$ is continuous in $[a,b]$. A polynomial $P(x)$ of degree n is the best uniform approximation to f if and only if there exist $n+2$ points $a \le x_0 < x_1 < \cdots < x_{n+1} \le b$ such that*

$$f(x_i) - P(x_i) = e(-1)^i \Delta, i = 0, 1, \ldots, n+1$$

where $e \in \{-1, 1\}$ and $\Delta := \max_{x\in[a,b]} |f(x) - P(x)|$.

We mostly use it in the following form. Suppose a polynomial $P(x)$ alternatively takes values $\pm\Delta$ in some $n+2$ points $a \le x_0 < x_1 < \cdots < x_{n+1} \le b$. Then for any other polynomial Q of degree n there exists a point $\xi \in [a,b]$ such that $|f(\xi) - Q(\xi)| \ge \Delta$.

Proof. One direction. Suppose, on the contrary, there exists p_n with $\max_{x\in[a,b]}|f(x)-p_n(x)| < \Delta$. Then $q_n(x) - p_n(x)$ has a root in each interval $(x_i, x_{i+1})\,, i = 0, 1, \ldots, n$, hence $q_n - p_n$ (of degree n) has at least $n+1$ different zeroes, therefore $q_n \equiv p_n$, a contradiction $\square$

Definition. *Chebyshev polynomial $T_n(x)$ of degree n is a polynomial for which $T_n(\cos\theta) = \cos n\theta$.*

All roots of $T_n(x)$ are in $[-1,1], x_k = cos\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right)$. T_n attains its extrema in $[-1,1]$ at the points $t_i := \cos\frac{k\pi}{n}, k = 0, 1, \ldots, n$ where it alternates $-1, 1, \ldots$.

Property 3.1. *(Extremal property of Chebyshev's polynomials) Given a real polynomial $P(x) = 2^{n-1}x^n + a_{n-1}x^{n-1} + ... + a_0$. Prove that there exists c from $[-1;1]$ such that $|P(c)| \geq 1$. The equality is attained in case $P \equiv T_n$.*

Suppose on the contrary, $|P(x)| < 1, \forall x \in [-1,1]$. Consider $Q(x) := T_n(x) - P(x)$. It's a polynomial of degree $n-1$ and $Q(x)$ changes its sign alternatively at the points $x_0, x_1, \ldots, x_n$ as $+, -, +, \ldots$ or $-, +, -, \ldots$ depending on the parity of n. It means that it vanishes in some points $\xi_j \in (x_j, x_{j+1}), j = 0, 1, \ldots, n-1$. So, $Q(x)$ is of degree $n-1$ and vanishes at n distinct points, hence $Q(x) \equiv 0$, contradiction.

Property 3.2. *Let $P(x)$ be a polynomial of degree n with $|P(x)| \leq 1, \forall x \in [-1,1]$ and $T_n(x)$ be the Chebyshev polynomial of degree n. Then*

$$|P(x)| \leq |T_n(x)|, \forall x \notin [-1,1].$$

This claim allows us when knowing the behaviour of a polynomial in some interval (i.e. its maximal absolute value) to predict its magnitude outside that interval. I think, it was first discovered by P. Chebyshev, but not quite sure. Although it may seem to involve hard calculations, in fact its proof is short and pleasant.

Proof. Suppose on the contrary there exists $x' \notin [-1,1]$ such that $|P(x')| > |T_n(x')|$. Let $x_k := \cos\left(\frac{k\pi}{n}\right), k = 0, 1, \ldots, n$. Apparently $T_n(x_k) = \pm 1, k = 0, 1, \ldots, n$, moreover $T_n(x)$ takes alternatively values, $1, -1$ at the knots $1 = x_0 > x_1 > \cdots > x_n = -1$.

We may assume $|P(x)| < 1$ in $[-1,1]$, otherwise we may consider $cP(x)$ instead of $P(x)$, where $0 < c < 1$ is close to 1 and such that $c|P(x')| > |T_n(x')|$ also holds. Suppose WLOG $x' > 1$. We also may assume $P(x') > T(x')$, otherwise we would take $-P(x)$. Consider now the polynomial $Q(x) := T_n(x) - P(x)$. It changes its sign alternatively at $x_0, x_1, \ldots, x_n$, hence there exist n points $\xi_k \in (x_k, x_{k+1}), k = 0, 1, \ldots, n-1$ where $Q(x)$ vanishes. There also exists a point ξ_n inside $(1, x')$ with $Q(\xi_n) = 0$, since $Q(1) > 0, Q(x') < 0$. Thus, we have found $n+1$ distinct roots of $Q(x)$ (which is of degree n) implying $P(x) \equiv T_n(x)$, a contradiction. $\square$

Problem 3.1. *(PFTB, handouts, etc.) Let $a_0, a_1, \ldots, a_n$ be real numbers such that $|a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0| \leq 1$ for all $x \in [-1, 1]$. Prove that $|a_0 x^n + a_1 x^{n-1} + \cdots + a_n| \leq 2^{n-1}$ for all $x \in [-1, 1]$*

Solution. Denote $P(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Applying Theorem 1 we get

$$P(\frac{1}{x}) \leq T_n(\frac{1}{x}), \forall x \in [-1, 1], x \neq 0.$$

It yields

$$|a_0 x^n + a_1 x^{n-1} + \cdots + a_n| \leq x^n T_n(\frac{1}{x}), \forall x \in [-1, 1]$$

It is well known that

$$T_n(y) = \frac{1}{2}\left(y - \sqrt{y^2 - 1}\right)^n + \frac{1}{2}\left(y + \sqrt{y^2 - 1}\right)^n$$

for $|y| \geq 1$. Hence

$$x^n T_n(\frac{1}{x}) = \frac{1}{2}\left(1 - \sqrt{1 - x^2}\right)^n + \frac{1}{2}\left(1 + \sqrt{1 - x^2}\right)^n, \forall x \in [-1, 1].$$

But $(1-t)^n + (1+t)^n$ attains its maximum in $[0, 1]$ when $t = 1$, thus the RHS of the above inequality is at most 2^{n-1} and the result follows. □

Definition. *Let $f(x)$ is continuous in $[a, b]$. A polynomial $P(x)$ of degree n is the best uniform approximation to f if*

$$E(Q) := \max_{x \in [a,b]} |f(x) - Q(x)|$$

attains its minimum value among all polynomials Q od degree n when $Q = P$.

Problem 3.2. *(IRAN NMO 2010) prove that for each natural number n there exist a polynomial with degree $2n + 1$ with coefficients in $\mathbb{Q}[x]$ such that it has exactly 2 complex zeros and it's irreducible in $\mathbb{Q}[x]$.*

Solution. Search for a polynomial $P(x)$ as $P(x) = \frac{1}{p}x^{2n+1} + T_{2n-1}(x) + \frac{1}{q}$, where: $T_m(x)$ is the Chebyshev plynomial of degree m and p, q are big enough prime numbers. $p \geq 2$ and big enough q ensure that $P(x)$ has at least $2n - 1$ real roots, big enough p will ensure that $P(x)$ will have exactly $2n-1$ real roots. Irreducibility of $P(x)$ in $\mathbb{Q}[x]$ follows by Eisenstein's criterion. □

Problem 3.3. *(India TST 2001 Day 5 Problem 3) Let $P(x)$ be a polynomial of degree n with real coefficients and let $a \geq 3$. Prove that*

$$\max_{0 \leq j \leq n+1} \left|a^j - P(j)\right| \geq 1$$

Solution. Let me outline a method, used very often, when dealing with polynomial approximation, either uniform in some interval, or at some discrete points, as in this particular case. Suppose, we have $n+2$ points $a \leq x_0 < \ldots < x_{n+1} \leq b$

in $[a, b]$ and corresponding real values $y_0, y_1, \dots, y_{n+1}$. We want to best approximate these values at these points with a polynomial with degree n, i.e. we are looking for a polynomial q_n with error of approximation $\max_{0\le i\le n+1} |q_n(x_i)-y_i|$ being minimal. When seeking to estimate the best possible approximation, the following simple claim is mostly applied.

Suppose we can find a polynomial p_n of degree n which oscillates around (x_i, y_i) with error at least Δ, i.e. $p_n(x_i) - y_i$, $i = 0, 1, \dots, n + 1$ changes alternatively its sign and

$$|p_n(x_i) - y_i| \ge \Delta \, , \; i = 0, 1, \dots, n + 1$$

Then, $\max_{0\le i\le n+1} |q_n(x_i) - y_i| \ge \Delta$ for any q_n of degree n.

Proof: Assume on the contrary, there exists q_n with $\max_{0\le i\le n+1} |q_n(x_i) - y_i| < \Delta$. Then $p_n - q_n$ has at least one root in each interval (x_i, x_{i+1}) , $i = 0, 1, \dots, n$, hence $p_n - q_n$, of degree at most n, has at least $n+1$ roots, implying $p_n \equiv q_n$, a contradiction $\square$

If in addition, the error at all x_i equals Δ, the constructed polynomial is the best possible approximation. This claim (or some version of it) is called Chebyshev equioscillation theorem. It's is very useful, since in order to get some lower bound of the best polynomial approximation, it's enough to construct a polynomial with good oscillating properties.

In our case $x_i = i$, $y_i = a^i$, $i = 0, 1, \dots, n + 1$. Let L be the (Lagrange) interpolation polynomial of degree $n+1$ with $L(x_i) = y_i$, $i = 0, 1, \dots, n+1$ and P be the interpolation polynomial (of degree $n + 1$) with $P(x_i) = (-1)^i$, $i = 0, \dots, n + 1$. Then $L + P$ oscillates around (x_i, y_i) with error $\Delta = 1$. The problem is that its degree is $n + 1$, not n as wanted. So, the trick is to consider $L(x) + C \cdot P(x)$, for appropriate constant C, such that to reduce the degree to n, that's we take $C = -a/b$, where a, b are the leading coefficients resp. of L and P. Thus, we obtain a polynomial with degree n, which oscillates at given points as needed, with error $\Delta = |a/b|$. It remains to estimate this error. One can use the fact that the leading coefficient of the interpolation polynomial through the points (x_i, y_i) is equal to $[y_0, y_1, \dots, y_{n+1}]$, where the last term means the divided difference at these points. One can check by induction that for $x_i = i, y_i = a^i, i = 0, 1, \dots, n + 1$ it holds

$$[y_0, y_1, \dots, y_{n+1}] = (a - 1)^{n+1}/(n + 1)!$$

and for $x_i = i, y_i = (-1)^i$, we have:

$$[y_0, y_1, \dots, y_{n+1}] = (2)^{n+1}/(n + 1)!$$

Therefore $C = -a/b = ((a - 1)/2)^{n+1} > 1$, hence applying the above considerations, the error of the best polynomial approximation of degree $n + 1$ to the points $x_i = 1, y_i = a^i$ is at least 1, or more precisely at least $((a - 1)/2)^{n+1}$. Moreover, since $L(x) + C \cdot P(x)$ oscillates with equal error at x_i, it's the best approximation (of degree n to the points (x_i, y_i)). $\square$

Problem 3.4. *(Iran NMO, 2011, p3) We define the polynomial $f(x)$ in $\mathbb{R}[x]$ as follows: $f(x) = x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0$. Prove that there exists $i \in \{1, 2, \ldots, n\}$ such that:*

$$|f(i)| \geq \frac{n!}{\binom{n}{i}}.$$

1-st solution. The idea is to find some polynomial of degree $n-2$ which oscillates around x^n at the knots $1, 2, \ldots, n$ with amplitudes not larger than $i(n-i)!, i = 1, 2, \ldots, n$. Suppose we managed to find such $P(x)$. Then, it's not possible to approximate x^n better than $P(x)$ at those knots. (with polynomials of degree $n-2$). Indeed, if P_1 is a better polynomial, then $P(x) - P_1(x)$ would change its sign alternatively at the knots, hence it has a root in each interval $(i, i+1)$, $i = 1, 2, \ldots, n-1$, that is, it has at least $n-1$ roots. However, it's of degree at most $n-2$, contradiction.

This idea can be implemented as follows. Construct a Lagrange interpolation polynomial $L(x)$ of degree $n-2$ such that $L(i) = i^n + (-1)^{n-1+i} i!(n-i)!, i = 1, 2, \ldots, n-1$. If it happened that $L(n) - n^n$ is of the same sign as $n!$ but with a larger magnitude, we are done. The same idea as in the Chebyshev's equioscillation theorem. The implementation follows.

We have,

$$\begin{aligned}
L(n) &= \sum_{i=1}^{n-1} \frac{(n-1)(n-2)\cdots(n-i+1)(n-i-1)\cdots 1}{(i-1)(i-2)\cdots(i-(i-1))(i-(i+1))\cdots(i-(n-1))} \left(i^n + (-1)^{n-1+i} i!(n-i)!\right) \\
&= \sum_{i=1}^{n-1} (-1)^{n-1-i} \frac{(n-1)!\left(i^n + (-1)^{n-1+i} i!(n-i)!\right)}{(n-i)(i-1)!(n-1-i)!} \\
&= \frac{1}{n} \sum_{i=1}^{n-1} (-1)^{n-1-i} \binom{n}{i} \left(i^{n+1} + (-1)^{n-1+i} i \cdot i!(n-i)!\right) \\
&= \frac{1}{n} n^{n+1} - \frac{1}{n} \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^{n+1} + \frac{1}{n} \sum_{i=1}^{n-1} (-1) \binom{n}{i} i \cdot i!(n-i)! \\
&= n^n - \frac{1}{n} \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^{n+1} + \frac{n!}{n} \sum_{i=1}^{n-1} i \\
&= n^n - \frac{1}{n} \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^{n+1} + n! \frac{n-1}{2} \\
&= n^n - \frac{1}{n} \frac{n(n+1)!}{2} + n! \frac{n-1}{2} \\
&= n^n - n!.
\end{aligned}$$

It remains to prove,

$$\frac{1}{n}\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}i^{n+1}=\frac{n(n+1)!}{2}.$$

□

2-nd solution. Assume on the contrary it's not true, i.e.

$$|f(i)|<\frac{n!}{\binom{n}{i}}=i!(n-i)!,\forall i\in[1..n] \qquad (1)$$

The idea is to take $n-1$-th finite difference with step 1 of $f(x)$ at $x=1$, i.e. $\Delta_1^{n-1}[f](x)$. One the one hand it can be estimated using (1). On the other hand it equals $\Delta_1^{n-1}[x^n](x=1)$, because $\Delta_h^{n-1}P(x)$ is zero for any polynomial P of degree $n-2$. These two estimates, as shown below, contradict each other. The details follow.

First, to remind that the finite difference $\Delta_h^k f(x)$ of order k and step h is defined as:

$$\Delta_h^k f(x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+jh).$$

In case $h=1$, we just write $\Delta^k f(x)$. Applying it to our function f and using (1) we get

$$\left|\Delta^{n-1}f(x)(1)\right|<\sum_{j=0}^{n-1}\binom{n-1}{j}j!(n-j)!=\frac{(n+1)!}{2} \qquad (2).$$

On the other side $\Delta^{n-1}x^n=n!x+n!\frac{n-1}{2}$ (as shown before). Hence, $\Delta^{n-1}[x^n](1)=\frac{(n+1)!}{2}$ and

$$\Delta f^{n-1}f(x)(1)=\frac{(n+1)!}{2}.$$

which contradicts with (2). □

4 Miscellaneous.

Problem 4.1. *(A Romanian TST, 2004) Given an integer $n\geq 1$, consider n distinct unit vectors in the plane, which have the common origin at point O. Suppose further that for some non-negative integer $m<n/2$, on either side of any line passing through O there are at least m of these vectors. Prove that the length of the sum of all n vectors cannot exceed $n-2m$.*

Solution. Denote by r the magnitude of the sum of the vectors. We'll prove a equivalent version of this statement, but m and r will swap their places, so it boils down to prove that given n distinct unit vectors in the plane with a common origin at O and with sum of magnitude r, there exists a line passing through O such that at most $n/2-r/2$ of these vectors lie on one side. The other side will contain at least $n/2+r/2$ vectors. That is the difference (discrepancy) between the vectors on the two sides is at least r.

Let us denote the vectors as $v_1, v_2, \ldots, v_n$. Without loss of generality, suppose $v_0 := \sum_{i=1}^n v_i$ has coordinates $(r, 0)$. For a vector $\mathbf{v}$ with origin at the point O, let φ, $\varphi \in [-\pi, \pi)$ be the angle between $\mathbf{v}$ and Ox-axis. We call φ the argument of $\mathbf{v}$. Let θ_i be the argument of $v_i, i = 1, 2, \ldots, n$. We choose a random vector $\mathbf{u}$ with argument $\theta, \theta \in [-\pi/2, \pi/2)$ with probability density $p(\theta) := \frac{1}{2}\cos\theta$. Roughly speaking, it means that the probability $\mathbf{u}$ has an argument in $[\theta, \theta+\Delta\theta]$ equals $\frac{1}{2}\cos\theta\, d\theta$. It's a correct definition since

$$\int_{-\pi/2}^{\pi/2} p(\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{2}\, d\theta = 1.$$

Let X_i be the random variable (indicator) that takes value 1 if the angle between v_i and $\mathbf{u}$ is acute; value -1 if this angle is obtuse; and value 0 if the angle is $\pm\pi/2$. Note that

$$\mathbb{P}(X_i = 1) = \frac{1}{2}\int_{-\pi/2+\theta_i}^{\pi/2} \cos\theta\, d\theta$$

$$\mathbb{P}(X_i = -1) = \frac{1}{2}\int_{-\pi/2}^{-\pi/2+\theta_i} \cos\theta\, d\theta.$$

Consider the random variable

$$X := \sum_{i=1}^{n} X_i.$$

Thus, X counts the difference between the number of vectors that are in the two half planes determined by the line through O and orthogonal to $\mathbf{u}$. Our goal is to prove $\mathbb{E}[X] = r$. We have,

$$\mathbb{E}[X_i] = \frac{1}{2}\int_{-\pi/2+\theta_i}^{\pi/2} \cos\theta\, d\theta - \frac{1}{2}\int_{-\pi/2}^{-\pi/2+\theta_i} \cos\theta\, d\theta.$$

It easily follows that $\mathbb{E}[X_i] = \cos\theta_i$. Using linearity of expectation, we obtain,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \cos\theta_i.$$

Note that, the projection of $\mathbf{v}_0$ onto Ox which is equal to r, on the other hand is equal to $\sum_{i=1}^n \cos\theta_i$. Hence,

$$\mathbb{E}[X] = r.$$

Therefore, there is a line through O such that the difference between the number of vectors among $v_1, v_2, \ldots, v_n$ that are inside one half-plane and the other half-plane respectively, is at least r. This completes the proof. □

Problem 4.2. *(Romania TST 1 2012, Problem 4) Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.*

Proof. Apply Nash-Williams theorem. □

Problem 4.3. *(Korean TST, p8) Problem (Korean TST, p8). Prove that for any simple planar graph G), one can remove the edges of a forest from $E(G)$ so that the remaining graph is 2-degenerate.*

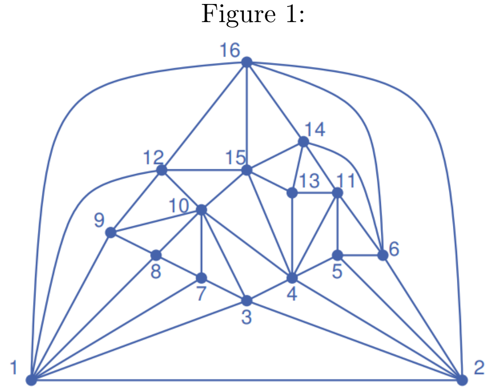
A graph is k-degenerate if its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ so that for each $i = 2, 3, \ldots, n$ the vertex v_i is connected to at most k vertices among the vertices $v_1, v_2, \ldots, v_{i-1}$.

Solution. We can assume G is near-triangular, otherwise would add some extra edges. Assume $v_1, v_2, \ldots, v_n$ is some ordering of the vertices of G. We denote by G_i the induced graph on $v_1, v_2, \ldots, v_i$, $i = 1, 2, \ldots, n$ and let C_i be the bounding cycle of the external face of G_i. This ordering of the vertices of G is called **canonical ordering** if for each $i = 3, 4, \ldots, n$ the following 3 conditions hold.

1. G_i is 2-connected and internally triangulated.
2. C_i contains the edge v_1v_2.
3. For $i < n$, vertex v_{i+1} lies in the outer face of G_i and all neighbors of v_{i+1} that are in G_i lie on the cycle C_i in consecutive order.

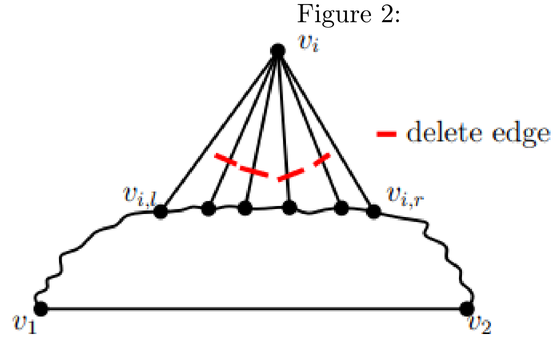
Every near-triangular graph has a canonical ordering - see a proof in the link. The next figure shows an example.

Figure 1:



We are now ready to give a solution to the original problem. If G is not fully triangulated we make it so, that is, we add extra edges to ensure that all faces of G, except possibly the external one, are triangles. Adding extra edges only makes the claim harder. So, we have a near-triangular graph and we can construct a canonical-ordering of its vertices as $v_1, v_2, \ldots, v_n$. Next, we begin the following procedure with the vertices, starting backwards, from v_n to going back to v_3 (see [2]) .We select the vertex v_i and consider its neighbors on the cycle C_{i-1}. They are the only neighbors that v_i has in G_{i-1} and they are consecutive on C_{i-1}, so we enumerate them as $u_1, u_2, \ldots, u_k$ in the order we go through the cycle C_{i-1} starting from v_1 and going around to v_2. Let $v_{i,\ell} := u_1$ be the "leftmost" one and $v_{i,r} := u_k$ - the "rightmost", see fig. 4.

Figure 2:



Next, we remove all the edges $v_iu_2, v_iu_3, \ldots, v_iu_{k-1}$, i.e., the edges incident with v_i that are between $v_iv_{i,\ell}$ and $v_iv_{i,r}$. Then we continue with the next vertex v_{i-1}. Finally, we come to v_3 and it is connected only to v_1 and v_2. Denote the set of edges that have been deleted as F and let the set of undeleted edges be U.

Let us now see that 1) F is a forest and 2) U is a 2-degenerate graph. The latter is obvious because in the enumeration $v_1, v_2, \ldots, v_n$ each vertex v_i (except v_1, v_2) is connected to exactly two vertices among $v_1, v_2, \ldots, v_{i-1}$, namely to $v_{i,\ell}$ and $v_{i,r}$. To see that F is a forest look again at fig. 2. The vertices $u_2, u_3, \ldots, u_{k-1}$ that are incident with the deleted edges are not connected to any other vertex among $v_n, v_{n-1}, \ldots, v_{i+1}$ (because G is planar). They are connected only to v_i and possibly to vertices that follow, i.e. $v_{i-1}, v_{i-2}, \ldots, v_1$. This means that F is a 1-degenerate graph because in the enumeration $v_n, v_{n-1}, \ldots, v_2$ each vertex is connected to at most one vertex before it by edges in F. This means that F is a forest. □

Problem 4.4. *(strategi-stealing argument) Let $\mathcal{F}$ be a family of subsets with 2 elements of some base set X. It is known that for any two elements $x, y \in X$ there exists a permutation π of the set X such that $\pi(x) = y$, and for any $A \in \mathcal{F}$*

$\pi(A) := \{\pi(a) : a \in A\} \in \mathcal{F}$.

A bear and crocodile play a game. At a move, a player paints one element of the set X in his own color: brown for the bear, green for the crocodile. The first player to fully paint one of the sets in $\mathcal{F}$ in his own color loses. If this does not happen and all the elements of X have been painted, it is a draw. The bear goes first. Prove that he doesn't have a winning strategy.

Solution. Assume for the sake of contradiction, the bear has a winning strategy. Note that for any $y \in X$ the bear also wins by painting at his first move y. Indeed, consider a permutation $\sigma, \sigma(x) = y$ that satisfies the condition in the statement. Assume that there are two bears that play on two different copies of X. The original (winning) bear plays on the first board and starts by painting x. The second bear paints y on the second board and waits for the crocodile's move. When the crocodile on the second board paints, say, z the second bear passes $\sigma^{-1}(z)$ painted in green to the first bear. The first bear plays x_1 and the second bear plays $\sigma(x_1)$ and so on.

Assume now, the bear starts with his winning strategy by painting an element $x_1 \in X$ brown. Let $\{x_1, y_1\} \in \mathcal{F}$. Suppose, the crocodile knows the winning strategy of the bear. He makes in his mind the following scenario. Assume that he, as the bear, starts by painting y_1 in green, and the bear (as the crocodile) responds by painting x_1 brown. Then the crocodile (as if he were the bear) in order to win plays by painting y_2 green. Now the crocodile acts. He paints y_2 and y_1 green, but the latter element only in his mind. In subsequent moves, he responds to the bear using bear's own strategy. Note that the bear cannot touch y_1 because if he paints it brown, he instantly loses since $\{x_1, y_1\} \in \mathcal{F}$. So, the bear cannot prevent the crocodile from using his own strategy and winning in the end. Contradiction! □

Problem 4.5. *(Russian TST 2018, day 4, p2) Let $\mathcal{F}$ be a finite family of subsets of some set X. It is known that for any two elements $x, y \in X$ there exists a permutation π of the set X such that $\pi(x) = y$, and*

$\pi(A) := \{\pi(a) : a \in A\} \in \mathcal{F}, \forall A \in \mathcal{F}$.

A bear and crocodile play a game. At a move, a player paints one or more elements of the set X in his own color: brown for the bear, green for the crocodile. The first player to fully paint one of the sets in $\mathcal{F}$ in his own color loses. If this does not happen and all the elements of X have been painted, it is a draw. The bear goes first. Prove that he doesn't have a winning strategy.

Solution. Assume on the contrary, the bear wins by painting at his first move a set $Y \subset X$. Of course, $Y \neq X$. Note that for any $x \in X, x \notin Y$, the bear also wins by painting at his first move some set $Y' \subset X, |Y'| = |Y|, x \in Y'$. Indeed, take any $y \in Y$ and consider a permutation $\sigma, \sigma(y) = x$ that satisfies the condition of the statement. Then, the bear also wins by painting at his first move the set $Y' := \sigma(Y)$. This can be shown by a symmetry argument as above. Assume that there are two bears that play on two different copies of X. The original (winning) bear plays on the first board and starts by painting Y brown. The second bear paints $Y' = \sigma(Y)$ brown and waits for the crocodile's move. After the crocodile responds with painting, say, a set Z', the second bear passes to the first bear the set $Z := \sigma^{-1}(Z')$ painted green. The first bear paints Y_1, according to his strategy. The second bear then paints $Y_1' := \sigma(Y_1)$ and so on.

So, now the bear with his winning strategy paints at first Y. The crocodile makes the following calculation in his mind. Assume, he knows the bear's strategy and he, as if he were the bear, starts by painting Y' green. Suppose the bear (as the crocodile) responds by painting the set $Y \setminus Y'$ brown. The crocodile (as the bear) must win and let his winning move in this situation be to paint (green) some set Y_1. Now, the crocodile acts. He paints in his first move the set $(Y' \setminus Y) \cup Y_1$ green. Then he follows the bear's winning strategy and wins because the only difference is that $Y' \cap Y$ is painted brown, instead of green, but if he wins when $Y' \cap Y$ is painted green, he will also win in this situation. Contradiction! $\square$